Problem 1.31

Solve the following differential equations:

(a) $y' = y/x + 1/y$;

(b) $y' = xy/(x^2 + y^2)$;

(c) $y' = x^2 + 2xy + y^2$;

(d) $yy'' = 2(y')^2$;

(e) $y' = (1 + x)y^2/x^2$;

(f) $x^2y' + xy + y^2 = 0$;

(g) $xy' = y(1 - \ln x + \ln y)$;

(h) $(x + y^2) + 2(y^2 + y + x - 1)y' = 0$, using an integrating factor of the form $I(x, y) = e^{ax+by}$;

(i) $-xy' + y = xy^2$ $[y(1) = 1]$;

(j) $y'' - (1 + x)^{-2}(y')^2 = 0$ $[y(0) = y'(0) = 1]$;

(k) $2xyy' + y^2 - x^2 = 0$;

(l) $y'' = (y')^2e^{-y}$ (if $y' = 1$ at $y = \infty$, find y' at $y = 0$);

(m) $y' = |y - x|$ [if $y(0) = \frac{1}{2}$, find $y(1)$];

(n) $xy' = y + xe^{y/x}$;

(o) $y' = (x^4 - 3x^2y^2 - y^3)/(2x^3y + 3y^2x)$;

(p) $(x^2 + y^2)y' = xy,\ y(e) = e$;

(q) $y'' + 2y'y = 0$ $[y(0) = y'(0) = -1]$;

(r) $x^2y'' + xy' - y = 3x^2$ $[y(1) = y(2) = 1]$;

(s) $y^3(y')^2y'' = -\frac{1}{2}$ $[y(0) = y'(0) = 1]$

(t) $xy' = y + \sqrt{xy}$;

(u) $(xy)y' + y \ln y = 2xy$ [try an integrating factor of the form $I = I(y)$]; [TYPO: The first term should be xy']

(v) $(x \sin y + e^y)y' = \cos y$;

(w) $(x + y^2x)y' + x^2y^3 = 0$ $[y(1) = 1]$;

(x) $(x - 1)(x - 2)y' + y = 2$ $[y(0) = 1]$;

(y) $y' = 1/(x + e^y)$;

(z) $xy' + y = y^2x^4$.

Solution

Part (a)

$$y' = y/x + 1/y$$

Multiply both sides of the ODE by y.

$$yy' = \frac{y^2}{x} + 1$$

Rewrite the left side as follows.

$$\frac{d}{dx}\left(\frac{1}{2}y^2\right) = \frac{y^2}{x} + 1$$

Bring the constant out of the derivative and move the y^2 term to the left.

$$\frac{1}{2}\frac{d}{dx}\left(y^2\right) - \frac{y^2}{x} = 1$$

Multiply both sides by 2 to get rid of the 1/2 factor.

$$\frac{d}{dx}(y^2) - \frac{2}{x}y^2 = 2$$

This is a first-order inhomogeneous ODE for y^2 that can be solved with an integrating factor I.

$$I = e^{\int^x -\frac{2}{s}\,ds} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

Multiply both sides of the equation by the integrating factor.

$$\frac{1}{x^2}\frac{d}{dx}(y^2) - \frac{2}{x^3}y^2 = \frac{2}{x^2}$$

The left side is now exact and can be written as $d/dx(Iy^2)$ as a result of the product rule.

$$\frac{d}{dx}\left(\frac{1}{x^2}y^2\right) = \frac{2}{x^2}$$

Integrate both sides with respect to x.

$$\frac{1}{x^2}y^2 = -\frac{2}{x} + C,$$

where C is an arbitrary constant. Multiply both sides by x^2.

$$y^2 = -2x + Cx^2$$

Therefore,

$$y(x) = \pm\sqrt{-2x + Cx^2}.$$

Part (b)

$$y' = xy/(x^2 + y^2)$$

Multiply the numerator and denominator on the right side by $1/x^2$.

$$y' = \frac{xy}{x^2 + y^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{\frac{y}{x}}{1 + \frac{y^2}{x^2}} = \frac{\frac{y}{x}}{1 + \left(\frac{y}{x}\right)^2}$$

The right-hand side suggests the substitution,

$$u = \frac{y}{x} \quad \to \quad xu = y$$

$$u + x\frac{du}{dx} = \frac{dy}{dx}.$$

The ODE is transformed to

$$u + x\frac{du}{dx} = \frac{u}{1 + u^2}.$$

Bring u to the right side.

$$x\frac{du}{dx} = -\frac{u^3}{1 + u^2}$$

This ODE can be solved by separation of variables.

$$\frac{1 + u^2}{u^3}\, du = -\frac{dx}{x}$$

Integrate both sides.

$$\int (u^{-3} + u^{-1})\, du = -\ln|x| + C$$

$$\frac{1}{-2}u^{-2} + \ln|u| = -\ln|x| + C$$

Bring $\ln|x|$ to the left and combine it with $\ln|u|$.

$$-\frac{1}{2}\frac{1}{u^2} + \ln|xu| = C$$

Now that the integration is done, change back to the original variable y.

$$-\frac{1}{2}\frac{x^2}{y^2} + \ln|y| = C$$

Multiply both sides by -2 and change the arbitrary constant. Therefore, the solution is expressed implicitly as

$$\frac{x^2}{y^2} - \ln y^2 = A.$$

Part (c)

$$y' = x^2 + 2xy + y^2$$

The right side is a perfect square.

$$y' = (x+y)^2$$

It suggests the substitution,

$$u = x + y \quad \rightarrow \quad u - x = y$$

$$\frac{du}{dx} - 1 = \frac{dy}{dx}$$

Plugging these into the ODE gives us

$$\frac{du}{dx} - 1 = u^2.$$

This equation can be solved by separation of variables.

$$\frac{du}{dx} = u^2 + 1$$

$$\frac{du}{u^2+1} = dx$$

Integrate both sides.

$$\arctan u = x + C$$

Take the tangent of both sides.

$$u(x) = \tan(x + C)$$

Now change back to the original variable y.

$$x + y = \tan(x + C)$$

Therefore,

$$y(x) = \tan(x + C) - x.$$

Part (d)

$$yy'' = 2(y')^2$$

Subtract $(y')^2$ from both sides.

$$yy'' - (y')^2 = (y')^2$$

Divide both sides by $(y')^2$.

$$\frac{yy'' - (y')^2}{(y')^2} = 1$$

Recognize that the left side is the derivative of a quotient.

$$\frac{d}{dx}\left(-\frac{y}{y'}\right) = 1$$

Integrate both sides with respect to x.

$$-\frac{y}{y'} = x + C_1$$

Multiply both sides by -1.

$$\frac{y}{y'} = -(x + C_1)$$

Invert both sides.

$$\frac{y'}{y} = -\frac{1}{x + C_1}$$

This ODE can be solved with separation of variables.

$$\frac{dy}{y} = -\frac{dx}{x + C_1}$$

Integrate both sides.

$$\ln|y| = -\ln|x + C_1| + C_2$$

Exponentiate both sides.

$$e^{\ln|y|} = e^{\ln|x+C_1|^{-1}+C_2}$$

$$|y| = \frac{e^{C_2}}{|x + C_1|}$$

Remove the absolute value sign on the left by introducing $\pm$ on the right side.

$$y(x) = \frac{\pm e^{C_2}}{|x + C_1|}$$

Use new arbitrary constants on the right side, A and B, and drop the absolute value sign—we can do this because A is arbitrary. Therefore,

$$y(x) = \frac{A}{x + B}.$$

Part (e)

$$y' = (1+x)y^2/x^2$$

This ODE can be solved by separation of variables.

$$\frac{dy}{dx} = \frac{1+x}{x^2}y^2$$

Split up the fraction on the right side with x.

$$\frac{dy}{y^2} = \left(\frac{1}{x^2} + \frac{1}{x}\right) dx$$

Integrate both sides.

$$-\frac{1}{y} = -\frac{1}{x} + \ln|x| + C$$

Combine the terms on the right side.

$$-\frac{1}{y} = \frac{-1 + x\ln|x| + Cx}{x}$$

Invert both sides and multiply both sides by -1.

$$y = \frac{x}{1 - x\ln|x| - Cx}$$

Introduce a new arbitrary constant A to eliminate the minus sign. Therefore,

$$y(x) = \frac{x}{1 - x\ln|x| + Ax}.$$

Part (f)

$$x^2y' + xy + y^2 = 0$$

This is a Bernoulli equation, so we start by dividing both sides by y^2.

$$x^2y^{-2}y' + xy^{-1} + 1 = 0$$

Now make the substitution,

$$u = y^{-1}$$
$$\frac{du}{dx} = -y^{-2}\frac{dy}{dx} \quad \rightarrow \quad -\frac{du}{dx} = y^{-2}\frac{dy}{dx}$$

Plug these into the ODE.

$$x^2\left(-\frac{du}{dx}\right) + xu + 1 = 0$$

Divide both sides by $-x^2$.

$$\frac{du}{dx} - \frac{1}{x}u - \frac{1}{x^2} = 0$$

Bring $1/x^2$ to the right side.

$$\frac{du}{dx} - \frac{1}{x}u = \frac{1}{x^2}$$

This is a first-order inhomogeneous ODE that can be solved by multiplying both sides by an integrating factor.

$$I = e^{\int^x -\frac{1}{s}\,ds} = e^{-\ln x} = x^{-1}$$

Proceed with the multiplication of both sides by I.

$$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = \frac{1}{x^3}$$

The left side is now exact and can be written as $d/dx(Iu)$ as a result of the product rule.

$$\frac{d}{dx}\left(\frac{1}{x}u\right) = \frac{1}{x^3}$$

Integrate both sides with respect to x.

$$\frac{1}{x}u = -\frac{1}{2x^2} + C$$

Multiply both sides by x to solve for u.

$$u(x) = -\frac{1}{2x} + Cx$$

Now that the integration is done, change back to the original variable y.

$$\frac{1}{y} = -\frac{1}{2x} + Cx$$

Combine the terms on the right side and use a new constant A for $2C$.

$$\frac{1}{y} = \frac{-1 + 2Cx^2}{2x} \quad \rightarrow \quad y(x) = \frac{2x}{Ax^2 - 1}$$

Part (g)

$$xy' = y(1 - \ln x + \ln y)$$

Divide both sides by x and combine the logarithms on the right side.

$$y' = \frac{y}{x}\left(1 - \ln \frac{y}{x}\right)$$

The right side suggests the subsitution,

$$u = \frac{y}{x} \quad \to \quad xu = y$$
$$u + x\frac{du}{dx} = \frac{dy}{dx}.$$

Plug these expressions into the ODE.

$$u + x\frac{du}{dx} = u(1 - \ln u)$$

Subtract u from both sides.

$$x\frac{du}{dx} = -u \ln u$$

This ODE can be solved by separation of variables.

$$\frac{du}{u \ln u} = -\frac{dx}{x}$$

Integrate both sides.

$$\int \frac{du}{u \ln u} = -\ln |x| + C$$

Use the following substitution to evaluate the integral on the left.

$$v = \ln u$$
$$dv = \frac{du}{u}$$

The integral becomes

$$\int \frac{dv}{v} = -\ln |x| + C.$$

So we have

$$\ln |v| = -\ln |x| + C.$$

Exponentiate both sides.

$$|v| = |x|^{-1} e^C$$

Introduce $\pm$ on the right side to eliminate the absolute value sign on the left.

$$v = \frac{\pm e^C}{|x|}$$

Use a new arbitrary constant A.

$$v = \frac{A}{|x|}$$

It's because A is arbitrary that we can drop the absolute value sign in the denominator. Change back to the variable u.

$$\ln u = \frac{A}{x}$$

Exponentiate both sides.

$$u = e^{A/x}$$

Now change back to the original variable y.

$$\frac{y}{x} = e^{A/x}$$

Multiply both sides by x to solve for y. Therefore,

$$y(x) = xe^{A/x}.$$

Part (h)

$(x+y^2)+2(y^2+y+x-1)y'=0$, using an integrating factor of the form $I(x,y)=e^{ax+by}$

This differential equation is of the form,

$$M(x,y)+N(x,y)\frac{dy}{dx}=0.$$

Multiplying both sides by an integrating factor $I(x,y)$ gives

$$I(x,y)M(x,y)+I(x,y)N(x,y)\frac{dy}{dx}=0. \tag{1}$$

Our aim is to determine the constants, a and b, in the provided function so that

$$\frac{\partial}{\partial y}I(x,y)M(x,y)=\frac{\partial}{\partial x}I(x,y)N(x,y).$$

This is the condition that has to hold in order for the ODE to be exact. Using the product rule, we have for the left side

$$\begin{aligned}\frac{\partial}{\partial y}I(x,y)M(x,y)&=\frac{\partial}{\partial y}(x+y^2)e^{ax+by}\\&=2ye^{ax+by}+(x+y^2)be^{ax+by}\\&=[2y+b(x+y^2)]e^{ax+by}.\end{aligned}$$

Using the product rule, we have for the right side

$$\begin{aligned}\frac{\partial}{\partial x}I(x,y)N(x,y)&=\frac{\partial}{\partial x}2(y^2+y+x-1)e^{ax+by}\\&=2e^{ax+by}+2(y^2+y+x-1)ae^{ax+by}\\&=2[1+a(y^2+y+x-1)]e^{ax+by}.\end{aligned}$$

In order for these partial derivatives to be equal, we require that

$$2y+b(x+y^2)=2[1+a(y^2+y+x-1)].$$

Expand both sides of the equation.

$$2y+bx+by^2=2+2ay^2+2ay+2ax-2a$$

This equation can only be true if we set $a=1$ and $b=2$. Thus, our integrating factor is $I(x,y)=e^{x+2y}$. The ODE we started with becomes exact as a result of multiplying both sides by this integrating factor. The fact that it is exact means there exists a potential function $\phi(x,y)$ such that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=I(x,y)M(x,y)\\\frac{\partial\phi}{\partial y}&=I(x,y)N(x,y).\end{aligned}$$

The ODE in equation (1) can hence be written as

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0. \tag{2}$$

Recall that for a function of two variables $\phi(x, y)$, its differential is defined as

$$d\phi = \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy.$$

Dividing both sides by dx yields the relationship between the total derivative of a function and its partial derivatives.

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

So equation (2) reduces to

$$\frac{d\phi}{dx} = 0.$$

Integrating both sides with respect to x gives

$$\phi(x, y) = A,$$

where A is an arbitrary constant. Our goal now is to find this potential function.

$$\frac{\partial \phi}{\partial x} = (x + y^2)e^{x+2y} \tag{3}$$

$$\frac{\partial \phi}{\partial y} = 2(y^2 + y + x - 1)e^{x+2y} \tag{4}$$

Since equation (3) looks simpler, integrate both sides of it partially with respect to x to solve for ϕ. Note that we would arrive at the same answer if we integrated both sides of equation (4) partially with respect to y.

$$\begin{aligned}
\phi(x, y) &= \int^x \left.\frac{\partial \phi}{\partial x}\right|_{x=s} ds + f(y) \\
&= \int^x (s + y^2)e^{s+2y}\, ds + f(y) \\
&= \int^x (se^s e^{2y} + y^2 e^s e^{2y})\, ds + f(y) \\
&= e^{2y}\int^x se^s\, ds + y^2 e^{2y}\int^x e^s\, ds + f(y) \\
&= e^{2y}(x - 1)e^x + y^2 e^{2y} e^x + f(y) \\
&= (x - 1 + y^2)e^{x+2y} + f(y),
\end{aligned}$$

where $f(y)$ is an arbitrary function. To determine it, differentiate $\phi(x, y)$ with respect to y.

$$\frac{\partial \phi}{\partial y} = 2(y^2 + y + x - 1)e^{x+2y} + f'(y)$$

In order for this equation to be consistent with equation (4), we require that $f'(y) = 0$, which means $f(y) = B$, a constant. Consequently,

$$\phi(x, y) = (x - 1 + y^2)e^{x+2y} + B.$$

So for the general solution to the ODE, we have

$$(x - 1 + y^2)e^{x+2y} + B = A.$$

Subtract B from both sides and introduce a new arbitrary constant C. Therefore,

$$(x - 1 + y^2)e^{x+2y} = C.$$

Part (i)

$$-xy' + y = xy^2 \; [y(1) = 1]$$

This is a Bernoulli equation. First get it into standard form by dividing both sides by $-x$.

$$y' - \frac{1}{x}y = -y^2$$

Divide both sides now by y^2.

$$y^{-2}y' - \frac{1}{x}y^{-1} = -1$$

Make the substitution,

$$u = y^{-1}$$
$$\frac{du}{dx} = -y^{-2}\frac{dy}{dx} \quad \rightarrow \quad -\frac{du}{dx} = y^{-2}\frac{dy}{dx}$$

Plug these expressions into the ODE.

$$-\frac{du}{dx} - \frac{1}{x}u = -1$$

Multiply both sides by -1.

$$\frac{du}{dx} + \frac{1}{x}u = 1$$

This is a first-order inhomogeneous equation that can be solved by multiplying both sides by an integrating factor I.

$$I = e^{\int^x \frac{1}{s}\,ds} = e^{\ln x} = x$$

Proceed with the multiplication.

$$x\frac{du}{dx} + u = x$$

The left side is now exact and can be written as $d/dx(Iu)$ as a result of the product rule.

$$\frac{d}{dx}(xu) = x$$

Integrate both sides of the equations with respect to x.

$$xu = \frac{1}{2}x^2 + C$$

Divide both sides by x to solve for u.

$$u(x) = \frac{1}{2}x + \frac{C}{x}$$

Now that the integration is done, change back to the original variable y.

$$\frac{1}{y} = \frac{1}{2}x + \frac{C}{x}$$

Write the right side as one term by combining the fractions.

$$\frac{1}{y} = \frac{x^2 + 2C}{2x}$$

Invert both sides to solve for y.

$$y(x) = \frac{2x}{x^2 + 2C}$$

Now that we have the general solution we can apply the initial condition to determine the constant in the denominator.

$$y(1) = \frac{2}{1 + 2C} = 1$$

Solving this equation yields $C = 1/2$. Therefore,

$$y(x) = \frac{2x}{x^2 + 1}.$$

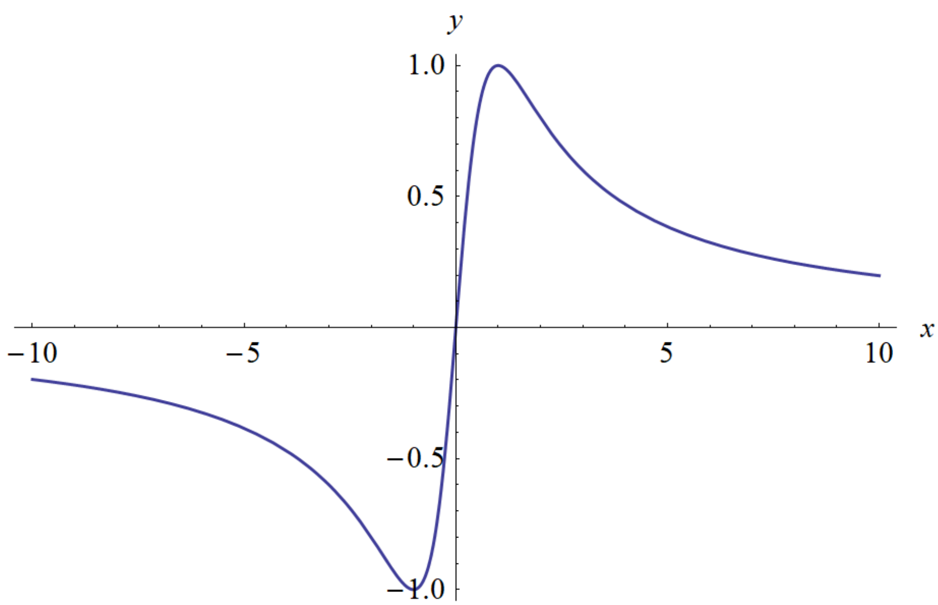

Figure 1: Plot of the solution for $-10 < x < 10$.

Part (j)

$$y'' - (1+x)^{-2}(y')^2 = 0 \; [y(0) = y'(0) = 1]$$

This ODE is first-order in y', so make the substitution,

$$u = y'$$
$$u' = y''.$$

Plugging these expressions into the ODE yields

$$u' - \frac{1}{(1+x)^2}u^2 = 0,$$

which can be solved by separation of variables. Bring the second term over to the right.

$$\frac{du}{dx} = \frac{1}{(1+x)^2}u^2$$

Separate variables.

$$\frac{du}{u^2} = \frac{dx}{(1+x)^2}$$

Integrate both sides.

$$-\frac{1}{u} = -\frac{1}{1+x} + C$$

Multiply both sides by -1 and combine the two terms on the right into one.

$$\frac{1}{u} = \frac{1 - C(1+x)}{1+x}$$

Invert both sides now to solve for u.

$$u(x) = \frac{1+x}{1 - C(1+x)}$$

Now that the integration is done, change back to the original variable y.

$$y' = \frac{1+x}{1 - C(1+x)}$$

At this point we can apply the first initial condition, $y'(0) = 1$, to determine C.

$$y'(0) = \frac{1}{1-C} = 1$$

Solving for C gives $C = 0$. So we have

$$y' = 1 + x.$$

Integrate both sides with respect to x to solve for y.

$$y(x) = x + \frac{1}{2}x^2 + D$$

Use the second initial condition, $y(0) = 1$, to determine D.

$$y(0) = D = 1$$

Therefore,

$$y(x) = x + \frac{1}{2}x^2 + 1.$$

Part (k)

$$2xyy' + y^2 - x^2 = 0$$

Rewrite the term with the derivative as follows.

$$x\frac{d}{dx}(y^2) + y^2 - x^2 = 0$$

Bring the x^2 term to the right.

$$x\frac{d}{dx}(y^2) + y^2 = x^2$$

Notice that the left side is exact and can be written as $d/dx(xy^2)$ as a result of the product rule.

$$\frac{d}{dx}(xy^2) = x^2$$

Integrate both sides with respect to x.

$$xy^2 = \frac{1}{3}x^3 + C$$

Divide both sides by x.

$$y^2 = \frac{1}{3}x^2 + \frac{C}{x}$$

Therefore,

$$y(x) = \pm\sqrt{\frac{1}{3}x^2 + \frac{C}{x}}.$$

Part (l)

$$y'' = (y')^2 e^{-y} \text{ (if } y' = 1 \text{ at } y = \infty\text{, find } y' \text{ at } y = 0)$$

Divide both sides by y'.

$$\frac{y''}{y'} = y' e^{-y}$$

Rewrite the left side as follows.

$$\frac{d}{dx} \ln y' = y' e^{-y}$$

Rewrite the right side as follows.

$$\frac{d}{dx} \ln y' = \frac{d}{dx}(-e^{-y})$$

Integrate both sides with respect to x.

$$\ln y' = -e^{-y} + C.$$

Exponentiate both sides.

$$y' = e^C e^{-e^{-y}}$$

Use a new arbitrary constant A.

$$y' = Ae^{-e^{-y}} \tag{1}$$

Now that we solved for y' in terms of y, we can use the provided boundary condition to determine A. As $y \to \infty$, $e^{-y} \to 0$, so we have

$$\lim_{y\to\infty} y' = Ae^0 = A = 1.$$

Now that we know A, we can find y' when $y = 0$.

$$\lim_{y\to 0} y' = e^{-e^0}$$

Therefore, y' at $y = 0$ is equal to e^{-1}. The general solution for y can be obtained by separation of variables in equation (1).

$$e^{e^{-y}}\, dy = A\, dx$$

Integrate both sides.

$$\int^y e^{e^{-s}}\, ds = Ax + B$$

The solution is only implicit for y.

Part (m)

$$y' = |y - x| \text{ [if } y(0) = \frac{1}{2}, \text{ find } y(1)]$$

The right side prompts the substitution,

$$u = y - x$$

$$\frac{du}{dx} = \frac{dy}{dx} - 1 \quad \rightarrow \quad \frac{du}{dx} + 1 = \frac{dy}{dx}.$$

Plug these expressions into the ODE.

$$\frac{du}{dx} + 1 = |u|$$

Bring 1 to the right side.

$$\frac{du}{dx} = |u| - 1$$

The absolute value is defined as

$$\begin{cases} u & u > 0 \\ -u & u < 0, \end{cases}$$

so there are two cases to consider here.

Case I: $u > 0$

Here we consider the first case.

$$\frac{du}{dx} = u - 1$$

This equation can be solved with separation of variables.

$$\frac{du}{u - 1} = dx$$

Integrate both sides.

$$\ln|u - 1| = x + C$$

Exponentiate both sides.

$$|u - 1| = e^x e^C$$

Eliminate the absolute value sign by introducing $\pm$ on the right side.

$$u - 1 = \pm e^C e^x$$

Use a new arbitrary constant.

$$u - 1 = Ae^x$$

Bring 1 to the right side to solve for u.

$$u(x) = 1 + Ae^x, \quad u > 0$$

Change back now to the original variable y.

$$y - x = 1 + Ae^x$$

Thus, for the first case we have

$$y(x) = x + 1 + Ae^x, \quad y - x > 0.$$

Case II: $u < 0$

Here we consider the second case.

$$\frac{du}{dx} = -u - 1$$

This equation can be solved with separation of variables.

$$\frac{du}{u+1} = -dx$$

Integrate both sides.

$$\ln|u+1| = -x + C$$

Exponentiate both sides.

$$|u+1| = e^{-x}e^C$$

Eliminate the absolute value sign by introducing $\pm$ on the right side.

$$u + 1 = \pm e^C e^{-x}$$

Use a new arbitrary constant.

$$u + 1 = Be^{-x}$$

Bring 1 to the right side to solve for u.

$$u(x) = -1 + Be^{-x}, \quad u < 0$$

Change back now to the original variable y.

$$y - x = -1 + Be^{-x}$$

Thus, for the second case we have

$$y(x) = x - 1 + Be^{-x}, \quad y - x < 0.$$

Putting the results of these two cases together, we have for the general solution

$$y(x) = \begin{cases} x + 1 + Ae^x & y - x > 0 \\ x - 1 + Be^{-x} & y - x < 0 \end{cases}.$$

To determine one of the constants, we use the provided initial condition, $y(0) = \frac{1}{2}$. Since y is bigger than x, we apply it to the first case.

$$y(0) = 1 + A = \frac{1}{2} \quad \rightarrow \quad A = -\frac{1}{2}$$

The solution is now

$$y(x) = \begin{cases} x + 1 - \frac{1}{2}e^x & y - x > 0 \\ x - 1 + Be^{-x} & y - x < 0 \end{cases}.$$

To determine the second unknown constant, we require that the solution be continuous everywhere, that is, when $y - x = 0$, the two expressions for $y(x)$ must yield the same result. Bring x to the left side.

$$y - x = \begin{cases} 1 - \dfrac{1}{2}e^x = 0 \\ -1 + Be^{-x} = 0 \end{cases}$$

We have here a system of two equations for two unknowns, x and B. Solving the system gives us $x = \ln 2$ and $B = 2$. Therefore, the solution to the ODE is

$$y(x) = \begin{cases} x + 1 - \frac{1}{2}e^x & y - x > 0 \\ x - 1 + 2e^{-x} & y - x < 0 \end{cases}.$$

Although we have determined the constants, this equation is only implicit for $y(x)$. Our aim now is to write an explicit expression for y, that is, one that depends only on x. The interpretation of this solution is as follows: above the line $y = x$, we use the first expression for $y(x)$ and below the same line, we use the second expression for $y(x)$. What we have to do is graph the functions and find out for what values of x this occurs.

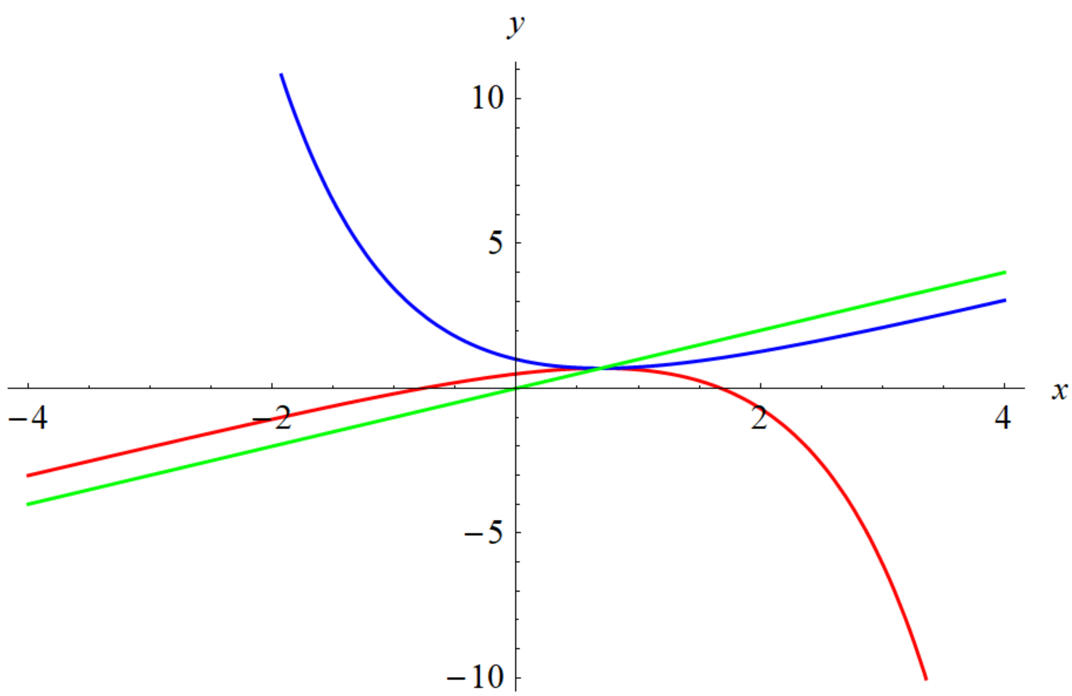

Figure 2: This is a plot of three functions for $-4 < x < 4$. The first expression for $y(x)$ is in red, the second expression for $y(x)$ is in blue, and the line, $y = x$, is in green.

As can be seen from the graph, the red line is above the green line to the left of the point of intersection, $x = \ln 2$. Also, the blue line is below the green line to the right of $x = \ln 2$. Therefore, the explicit solution for $y(x)$ is this.

$$y(x) = \begin{cases} x + 1 - \frac{1}{2}e^x & x < \ln 2 \\ x - 1 + 2e^{-x} & x \geq \ln 2 \end{cases}$$

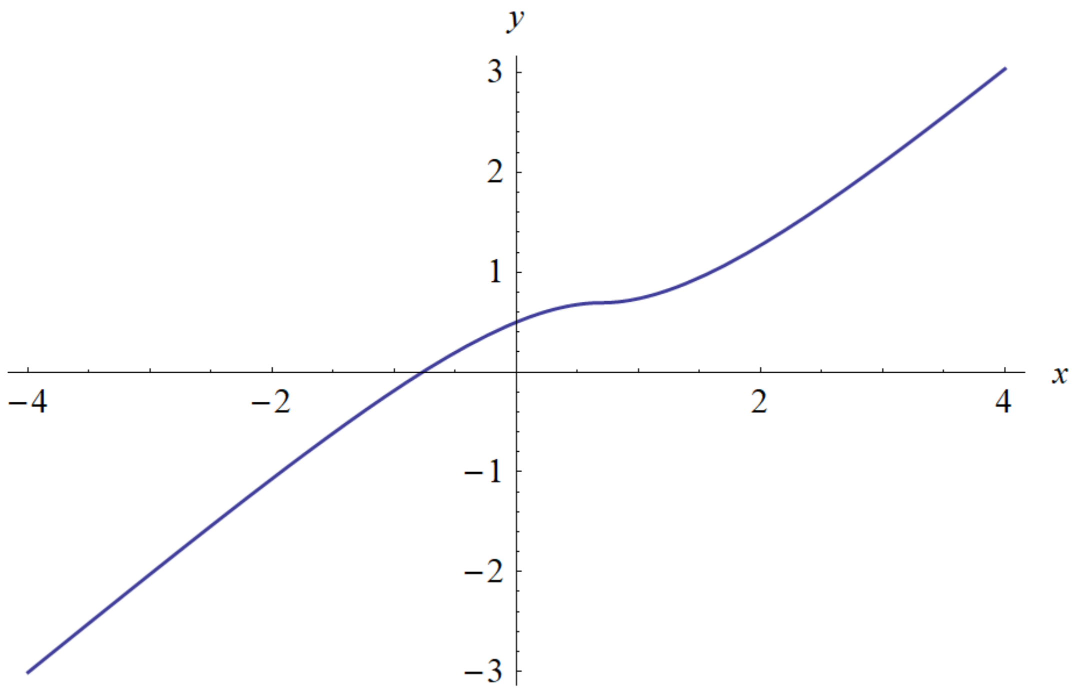

Figure 3: Plot of the solution for $-4 < x < 4$.

Finally, we are in a position to answer the question. Since $\ln 2 \approx 0.69$, we use the second expression to determine $y(1)$.

$$y(1) = 2e^{-1} \approx 0.73$$

Part (n)

$$xy' = y + xe^{y/x}$$

Divide both sides of the equation by x.

$$y' = \frac{y}{x} + e^{y/x}$$

The right side prompts the substitution,

$$u = \frac{y}{x} \quad \rightarrow \quad xu = y$$

$$u + x\frac{du}{dx} = \frac{dy}{dx}.$$

Plugging these expressions into the ODE, we have

$$u + x\frac{du}{dx} = u + e^u.$$

Cancel u from both sides.

$$x\frac{du}{dx} = e^u$$

This equation can be solved by separation of variables.

$$e^{-u}\,du = \frac{dx}{x}$$

Integrate both sides.

$$-e^{-u} = \ln|x| + C$$

Multiply both sides by -1.

$$e^{-u} = -\ln|x| - C$$

Take the logarithm of both sides.

$$-u = \ln(-\ln|x| - C)$$

Use a new arbitrary constant $\ln B$, remove the minus sign in front of the logarithm by inverting its argument, and multiply both sides by -1 to solve for u.

$$u(x) = -\ln\left(\ln\frac{1}{|x|} + \ln B\right)$$

Now that the integration is done, change back to the original variable y. Combine the logarithms and remove the minus sign in front of the logarithm by inverting its argument.

$$\frac{y}{x} = \ln\frac{1}{\ln\frac{B}{|x|}}$$

The point of using $\ln B$ for the new arbitrary constant is so that B is on top of the absolute value sign here. This allows us to drop the absolute value sign because it doesn't matter whether x is positive or negative. Multiply both sides by x to solve for y. Therefore,

$$y(x) = x\ln\frac{1}{\ln\frac{B}{x}}.$$

Part (o)

$$y' = (x^4 - 3x^2y^2 - y^3)/(2x^3y + 3y^2x)$$

Bring all terms over to the left side.

$$y^3 + 3x^2y^2 - x^4 + (2x^3y + 3y^2x)\frac{dy}{dx} = 0$$

This ODE is of the form,

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

Check to see whether $M_y = N_x$ or not. It it's not, then we'll have to multiply both sides by an integrating factor.

$$\frac{\partial M}{\partial y} = 3y^2 + 6x^2y$$
$$\frac{\partial N}{\partial x} = 6x^2y + 3y^2$$

$M_y = N_x$, so the ODE is exact. This implies that there exists a potential function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = M(x, y) \quad (1)$$
$$\frac{\partial \phi}{\partial y} = N(x, y). \quad (2)$$

The ODE thus becomes

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0.$$

Recall that the differential of a function of two variables $\phi(x, y)$ is

$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy.$$

Divide both sides by dx to obtain the relationship between the total derivative of $\phi(x, y)$ and the partial derivatives of $\phi(x, y)$.

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

Consequently, the ODE becomes

$$\frac{d\phi}{dx} = 0.$$

Integrate both sides with respect to x to obtain the solution to the ODE.

$$\phi(x, y) = A,$$

where A is an arbitrary constant. Our aim now is to determine $\phi(x, y)$ using equations (1) and (2).

$$\frac{\partial \phi}{\partial x} = y^3 + 3x^2y^2 - x^4 \quad (1)$$
$$\frac{\partial \phi}{\partial y} = 2x^3y + 3y^2x \quad (2)$$

Integrate the second equation partially with respect to y to solve for ϕ. Note that we could integrate the first equation partially with respect to x to solve for ϕ as well. We would get the same answer either way.

$$\begin{aligned}\phi(x,y) &= \int^y \left.\frac{\partial\phi}{\partial y}\right|_{y=s} ds + f(x) \\ &= \int^y (2x^3 s + 3s^2 x)\,ds + f(x) \\ &= \int^y 2x^3 s\,ds + \int^y 3s^2 x\,ds + f(x) \\ &= 2x^3 \int^y s\,ds + 3x \int^y s^2\,ds + f(x) \\ &= x^3y^2 + xy^3 + f(x)\end{aligned}$$

In order to determine the arbitrary function $f(x)$, we have to use equation (1). Differentiate the expression we just obtained with respect to x.

$$\frac{\partial\phi}{\partial x} = 3x^2y^2 + y^3 + f'(x)$$

Comparing this with equation (1), we see that $f'(x)$ has to be equal to $-x^4$ in order to be consistent. Hence, $f(x) = -x^5/5$. Therefore, the general solution to the ODE is

$$x^3y^2 + xy^3 - \frac{x^5}{5} = A.$$

Part (p)

$$(x^2 + y^2)y' = xy, \; y(e) = e$$

Divide both sides by $x^2 + y^2$ to solve for y'.

$$y' = \frac{xy}{x^2 + y^2}$$

Multiply the numerator and denominator on the right side by $1/x^2$.

$$y' = \frac{xy}{x^2 + y^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{\frac{y}{x}}{1 + \frac{y^2}{x^2}} = \frac{\frac{y}{x}}{1 + \left(\frac{y}{x}\right)^2}$$

The right-hand side suggests the substitution,

$$u = \frac{y}{x} \quad \to \quad xu = y$$

$$u + x\frac{du}{dx} = \frac{dy}{dx}.$$

The ODE is transformed to

$$u + x\frac{du}{dx} = \frac{u}{1 + u^2}.$$

Bring u to the right side.

$$x\frac{du}{dx} = -\frac{u^3}{1 + u^2}$$

This ODE can be solved by separation of variables.

$$\frac{1 + u^2}{u^3}\, du = -\frac{dx}{x}$$

Integrate both sides.

$$\int (u^{-3} + u^{-1})\, du = -\ln|x| + C$$

$$\frac{1}{-2}u^{-2} + \ln|u| = -\ln|x| + C$$

Bring $\ln|x|$ to the left and combine it with $\ln|u|$.

$$-\frac{1}{2}\frac{1}{u^2} + \ln|xu| = C$$

Now that the integration is done, change back to the original variable y.

$$-\frac{1}{2}\frac{x^2}{y^2} + \ln|y| = C$$

Multiply both sides by -2.

$$\frac{x^2}{y^2} - 2\ln\, y = -2C$$

We can determine $-2C$ by using the provided boundary condition, $y(e) = e$.

$$1 - 2\ln\, e = -2C \quad \to \quad -2C = -1$$

Therefore,

$$\frac{x^2}{y^2} - 2\ln\, y = -1.$$

Part (q)

$$y'' + 2y'y = 0 \; [y(0) = y'(0) = -1]$$

The left side of the ODE can be written as follows.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{d}{dx}(y^2) = 0$$

Integrate both sides with respect to x.

$$\frac{dy}{dx} + y^2 = A$$

We can determine A by using the provided initial conditions. When $x = 0$, y and dy/dx are equal to -1.

$$-1 + (-1)^2 = A \quad \rightarrow \quad A = 0,$$

so the ODE simplifies to

$$\frac{dy}{dx} + y^2 = 0.$$

Move y^2 over to the right side.

$$\frac{dy}{dx} = -y^2$$

This ODE can be solved by separation of variables.

$$y^{-2}\, dy = -dx$$

Integrate both sides.

$$-\frac{1}{y} = -x + B$$

Plug in the initial conditions once again to determine B.

$$1 = B$$

So we have

$$-\frac{1}{y} = -x + 1$$

Multiply both sides by -1.

$$\frac{1}{y} = x - 1$$

Invert both sides to solve for y. Therefore,

$$y(x) = \frac{1}{x-1}.$$

Part (r)

$$x^2y'' + xy' - y = 3x^2 \; [y(1) = y(2) = 1]$$

This is an inhomogeneous ODE, so the general solution is the sum of the complementary solution y_c and the particular solution y_p.

$$y(x) = y_c + y_p$$

We'll start by finding y_c, which is the solution to the associated homogeneous equation.

$$x^2y_c'' + xy_c' - y_c = 0$$

This ODE is equidimensional since the change in scale $x \to ax$ leaves the equation unchanged. Thus, the solution is of the form $y_c = x^r$. Our task now is to plug this expression into the ODE to determine the values of r for which it holds.

$$y_c = x^r \quad \to \quad y_c' = rx^{r-1} \quad \to \quad y_c'' = r(r-1)x^{r-2}$$

Substituting these expressions into the ODE yields

$$r(r-1)x^r + rx^r - x^r = 0.$$

Divide both sides by x^r to obtain the indicial equation.

$$r(r-1) + r - 1 = 0$$

r cancels out.

$$r^2 - 1 = 0$$

Factor the left side.

$$(r-1)(r+1) = 0$$

Thus, $r = 1$ or $r = -1$. We can now write the solution for the associated homogeneous equation.

$$y_c(x) = C_1x^1 + C_2x^{-1}$$

Our next goal is to determine the particular solution y_p. To do this, we will use the method of variation of parameters. That is, we will assume y_p has the form

$$y_p = u_1(x)x + u_2(x)x^{-1},$$

where u_1 and u_2 satisfy

$$xu_1' + x^{-1}u_2' = 0$$

$$u_1' + (-1)x^{-2}u_2' = \frac{3x^2}{x^2} = 3.$$

Solve this system of equations for u_1' and u_2' using Cramer's rule.

$$u_1' = \frac{\begin{vmatrix} 0 & x^{-1} \\ 3 & -x^{-2} \end{vmatrix}}{\begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix}} = \frac{-\frac{3}{x}}{-\frac{2}{x}} = \frac{3}{2}$$

$$u_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix}} = \frac{3x}{-\frac{2}{x}} = -\frac{3}{2}x^2$$

Now that we know u_1' and u_2', we can determine u_1 and u_2 by integration. We're not concerned with the integration constants.

$$u_1(x) = \frac{3}{2}x$$
$$u_2(x) = -\frac{1}{2}x^3$$

Hence, the particular solution is

$$y_p = \frac{3}{2}x^2 - \frac{1}{2}x^2 = x^2.$$

Therefore, the general solution is

$$y(x) = C_1 x + C_2 x^{-1} + x^2.$$

We can now determine the two arbitrary constants, C_1 and C_2, by applying the provided boundary conditions, $y(1) = 1$ and $y(2) = 1$. The result is the following system of equations.

$$y(1) = C_1 + C_2 + 1 = 1$$
$$y(2) = 2C_1 + \frac{C_2}{2} + 4 = 1$$

Solving the system gives us $C_1 = -2$ and $C_2 = 2$. Therefore,

$$y(x) = -2x + \frac{2}{x} + x^2.$$

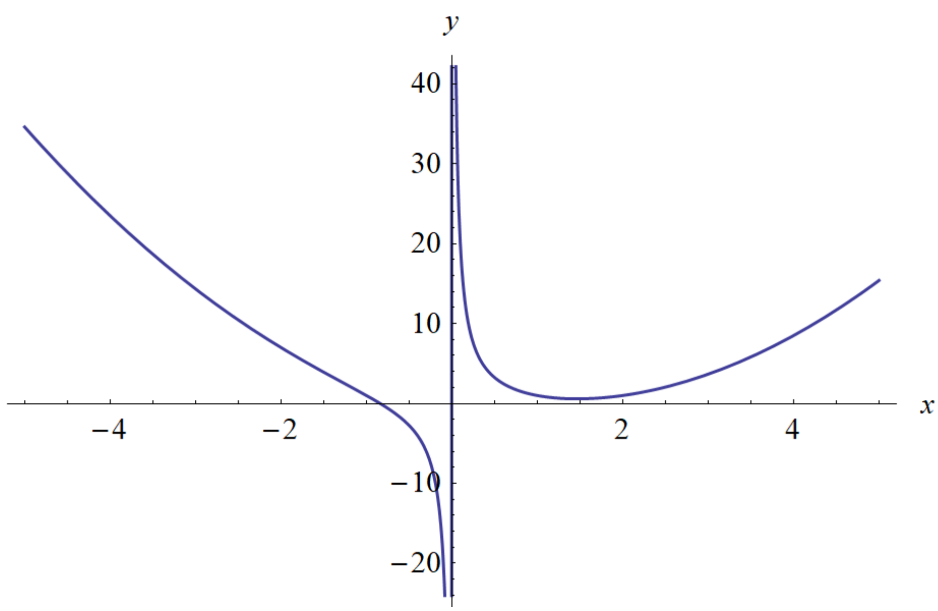

Figure 4: Plot of the solution for $-5 < x < 5$.

Part (s)

$$y^3(y')^2y'' = -\frac{1}{2} \; [y(0) = y'(0) = 1]$$

This ODE is second-order and autonomous, meaning the independent variable x does not appear in the equation. We can hence make the substitution,

$$\begin{aligned} y'(x) &= u(y) \\ y''(x) &= \frac{du}{dy}\frac{dy}{dx} = u'(y)u(y), \end{aligned}$$

to reduce the equation's order and make it easier to solve. Plugging these expressions into the ODE gives us

$$y^3u^2u'u = -\frac{1}{2},$$

which can be solved by separation of variables.

$$y^3u^3\frac{du}{dy} = -\frac{1}{2}$$

Separate variables.

$$u^3\,du = -\frac{1}{2}y^{-3}\,dy$$

Integrate both sides.

$$\frac{1}{4}u^4 = \frac{1}{4}y^{-2} + \frac{C}{4}$$

Multiply both sides by 4.

$$u^4 = \frac{1}{y^2} + C$$

Take the fourth root of both sides to solve for u.

$$u(y) = \sqrt[4]{\frac{1}{y^2} + C}$$

Now that we have u, change back to the original variable y.

$$y'(x) = \sqrt[4]{\frac{1}{y^2} + C}$$

At this point, use the provided boundary conditions, $y(0) = 1$ and $y'(0) = 1$, to determine the integration constant C.

$$y'(0) = \sqrt[4]{\frac{1}{[y(0)]^2} + C} \quad \to \quad 1 = \sqrt[4]{1 + C} \quad \to \quad C = 0$$

The ODE has thus been simplified to

$$\frac{dy}{dx} = \sqrt[4]{\frac{1}{y^2}} = \frac{1}{y^{1/2}},$$

which can be solved by separation of variables.

$$y^{1/2}\,dy = dx$$

Integrate both sides.

$$\frac{2}{3}y^{3/2} = x + B$$

Use the boundary condition $y(0) = 1$ to determine B.

$$\frac{2}{3} = B$$

So we have

$$\frac{2}{3}y^{3/2} = x + \frac{2}{3}.$$

Multiply both sides by 3/2.

$$y^{3/2} = \frac{3}{2}x + 1$$

Raise both sides to the 2/3 power to solve for y. Therefore,

$$y(x) = \left(\frac{3}{2}x + 1\right)^{2/3}.$$

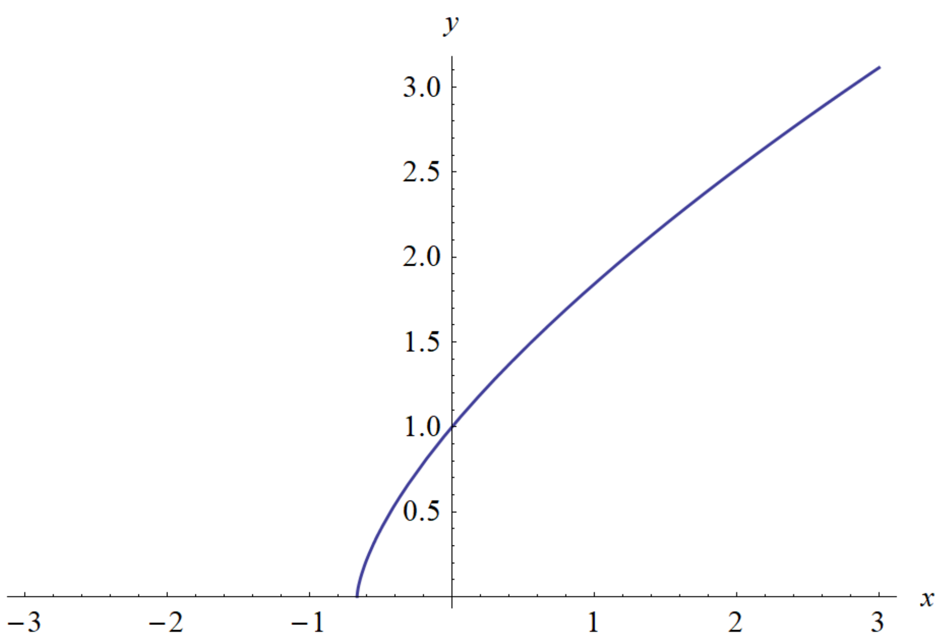

Figure 5: Plot of the solution for $-3 < x < 3$.

Part (t)

$$xy' = y + \sqrt{xy}$$

Divide both sides by x.

$$y' = \frac{y}{x} + \frac{1}{x}\sqrt{xy}$$

Bring x inside the square root.

$$y' = \frac{y}{x} + \sqrt{\frac{y}{x}}$$

The right side prompts the substitution,

$$u = \frac{y}{x} \quad \rightarrow \quad xu = y$$

$$u + x\frac{du}{dx} = \frac{dy}{dx},$$

Plugging these expressions into the ODE gives us

$$u + x\frac{du}{dx} = u + \sqrt{u}.$$

Cancelling u, we have here an ODE we can solve with separation of variables.

$$x\frac{du}{dx} = \sqrt{u}$$

Separate variables.

$$u^{-1/2}\,du = \frac{dx}{x}$$

Integrate both sides. Use $\ln C$ for the integration constant.

$$2u^{1/2} = \ln|x| + \ln C$$

Combine the logarithms.

$$2u^{1/2} = \ln C|x|$$

Because C is arbitrary, we can drop the absolute value sign. Divide both sides by 2.

$$u^{1/2} = \frac{1}{2}\ln Cx$$

Square both sides to solve for u.

$$u(x) = \frac{1}{4}(\ln Cx)^2$$

Change back now to the original variable y.

$$\frac{y}{x} = \frac{1}{4}(\ln Cx)^2$$

Multiply both sides by x to solve for y. Therefore,

$$y(x) = \frac{x}{4}(\ln Cx)^2.$$

Part (u)

$$(xy)y' + y\ln y = 2xy \text{ [try an integrating factor of the form } I = I(y)]$$

In order for an integrating factor of the form $I = I(y)$ to work, the ODE has to instead be

$$xy' + y\ln\, y = 2xy.$$

I confirmed this with one of the authors, Mr. Bender.

Solution by an Integrating Factor

Bring $2xy$ to the left side and factor y.

$$xy' + y(\ln\, y - 2x) = 0$$

Multiply both sides by the integrating factor $I(y)$.

$$xI(y)y' + yI(y)(\ln\, y - 2x) = 0$$

For this ODE to be exact, we require that

$$\frac{\partial}{\partial y}[yI(y)(\ln\, y - 2x)] = \frac{\partial}{\partial x}[xI(y)].$$

The right side is a function of y only. For the left side to be as well, $yI(y)$ must be equal to a constant. An appropriate integrating factor is thus

$$I(y) = \frac{1}{y}.$$

The ODE becomes

$$\frac{x}{y}y' + \ln\, y - 2x = 0,$$

which is exact. This means there exists a potential function $\phi = \phi(x, y)$ such that

$$\frac{\partial\phi}{\partial x} = \ln\, y - 2x \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \frac{x}{y}. \tag{2}$$

Substituting these into the ODE, we get

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0.$$

The differential of a function $\phi(x, y)$ is defined as

$$d\phi = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy.$$

Dividing both sides by dx gives the relationship between the total derivative of ϕ and the partial derivatives of ϕ.

$$\frac{d\phi}{dx} = \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx}$$

Hence, the ODE simplifies to

$$\frac{d\phi}{dx} = 0.$$

Integrate both sides with respect to x to obtain the general solution.

$$\phi = A,$$

where A is an arbitrary constant. Our task now is to determine the potential function ϕ from equations (1) and (2). Integrate equation (1) partially with respect to x.

$$\begin{aligned}\phi(x,y) &= \int^x \left.\frac{\partial\phi}{\partial x}\right|_{x=s} ds + f(y) \\ &= \int^x (\ln y - 2s)\, ds + f(y) \\ &= x\ln y - x^2 + f(y)\end{aligned}$$

To determine the arbitrary function $f(y)$, differentiate this expression partially with respect to y and compare it with equation (2).

$$\frac{\partial\phi}{\partial y} = \frac{x}{y} + f'(y)$$

We see that $f'(y)$ has to equal zero, which means $f(y) = B$, a constant. The potential function is consequently

$$\phi(x,y) = x\ln y - x^2 + B,$$

which means the general solution to the ODE is

$$x\ln y - x^2 + B = A.$$

Subtract B from both sides and use a new arbitrary constant C.

$$x\ln y - x^2 = C.$$

This equation can be solved for y explicitly. Bring x^2 to the right side.

$$x\ln y = C + x^2$$

Divide both sides by x.

$$\ln y = x + \frac{C}{x}$$

Exponentiate both sides to solve for y. Therefore,

$$y(x) = e^{x+C/x}.$$

Solution by a Substitution

$$xy' + y \ln y = 2xy$$

Divide both sides of the ODE by xy.

$$y^{-1}y' + \frac{1}{x} \ln y = 2$$

Make the substitution,

$$u = \ln y.$$

Take the derivative of both sides with respect to x to find out what y' is in terms of the new variable.

$$\frac{du}{dx} = y^{-1}\frac{dy}{dx}$$

Plug these expressions into the ODE.

$$\frac{du}{dx} + \frac{1}{x}u = 2$$

This is a first-order inhomogeneous equation that we can solve with an integrating factor I.

$$I = e^{\int^x \frac{1}{s}\, ds} = e^{\ln x} = x$$

Multiply both sides of the ODE by the integrating factor.

$$x\frac{du}{dx} + u = 2x$$

The left side can now be written as $d/dx(Iu)$ as a result of the product rule.

$$\frac{d}{dx}(xu) = 2x$$

Integrate both sides with respect to x.

$$xu = x^2 + C$$

Divide both sides by x to solve for u.

$$u(x) = x + \frac{C}{x}$$

Now that we have u, change back to the original variable y.

$$\ln y = x + \frac{C}{x}$$

Exponentiate both sides to solve for y. Therefore,

$$y(x) = e^{x+C/x}.$$

Part (v)

$$(x \sin y + e^y)y' = \cos y$$

Bring $\cos y$ to the left side.

$$-\cos y + (x \sin y + e^y)y' = 0$$

This ODE has the form,

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

Check to see whether $M_y = N_x$. If it's not, we'll have to use an integrating factor.

$$\frac{\partial M}{\partial y} = \sin y$$
$$\frac{\partial N}{\partial x} = \sin y$$

The two partial derivatives are equal, which means the ODE is exact. This implies that there exists a potential function $\phi(x, y)$ such that

$$\frac{\partial \phi}{\partial x} = M(x, y) \quad (1)$$
$$\frac{\partial \phi}{\partial y} = N(x, y). \quad (2)$$

Substituting these relations into the ODE gives

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0.$$

Recall that the differential of a function of two variabes, $\phi = \phi(x, y)$, is this.

$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy$$

Dividing both sides by dx gives us the relationship between the total derivative of ϕ and the partial derivatives of it.

$$\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx}$$

Substitution into the ODE reduces it to

$$\frac{d\phi}{dx} = 0.$$

Integrating both sides with respect to x gives the general solution.

$$\phi = A,$$

where A is an arbitrary constant. Our task now is to find this potential function $\phi(x, y)$ using equations (1) and (2).

$$\frac{\partial \phi}{\partial x} = -\cos y \quad (1)$$
$$\frac{\partial \phi}{\partial y} = x \sin y + e^y \quad (2)$$

We will solve for ϕ by integrating both sides of equation (1) partially with respect to x. Note that we would get the same answer for ϕ integrating both sides of equation (2) partially with respect to y.

$$\begin{aligned}\phi(x,y) &= \int^x \left.\frac{\partial\phi}{\partial x}\right|_{x=s} ds + f(y) \\ &= \int^x -\cos y\, ds + f(y) \\ &= -x\cos y + f(y)\end{aligned}$$

Differentiate this expression we just obtained with respect to y.

$$\frac{\partial\phi}{\partial y} = x\sin y + f'(y)$$

Comparing this result with equation (2), we see that $f'(y)$ has to be equal to e^y in order to be consistent, which means $f(y) = e^y + C$. We thus have

$$-x\cos y + e^y + C = A$$

for the general solution to the ODE. Bring C to the left and use a new arbitrary constant B. Therefore,

$$-x\cos y + e^y = B$$

is the general (albeit implicit) solution for $y(x)$.

Part (w)

$$(x + y^2 x)y' + x^2 y^3 = 0 \; [y(1) = 1]$$

This ODE can be solved by separation of variables.

$$x(1 + y^2)\frac{dy}{dx} + x^2 y^3 = 0$$

Bring $x^2 y^3$ over to the right.

$$x(1 + y^2)\frac{dy}{dx} = -x^2 y^3$$

Separate variables.

$$\frac{1 + y^2}{y^3}\, dy = -x\, dx$$

Integrate both sides.

$$\int^y \left(s^{-3} + \frac{1}{s}\right) ds = -\frac{1}{2}x^2 + C$$

Evaluate the integral on the left.

$$\frac{1}{-2}y^{-2} + \ln|y| = -\frac{1}{2}x^2 + C$$

Use the given boundary condition, $y(1) = 1$, to determine C.

$$-\frac{1}{2} = -\frac{1}{2} + C \quad \to \quad C = 0$$

So we have

$$\frac{1}{2}\left(x^2 - \frac{1}{y^2}\right) + \ln|y| = 0.$$

In order to obtain a single positive value of y when $x = 1$, we restrict the solution to positive values of y by dropping the absolute value sign.

$$\frac{1}{2}\left(x^2 - \frac{1}{y^2}\right) + \ln y = 0$$

Do note, though, that because we divided by x when we separated variables, the solution for y is not defined when $x = 0$. Therefore,

$$\frac{1}{2}\left(x^2 - \frac{1}{y^2}\right) + \ln y = 0, \quad x \neq 0.$$

Part (x)

$$(x-1)(x-2)y' + y = 2 \; [y(0) = 1]$$

Divide both sides $(x-1)(x-2)$ to isolate the y' term.

$$y' + \frac{1}{(x-1)(x-2)}y = \frac{2}{(x-1)(x-2)}$$

This is a first-order ODE that can be solved with an integrating factor I.

$$I = e^{\int^x \frac{1}{(s-1)(s-2)}\,ds}$$

To evaluate the integral, use partial fraction decomposition.

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

Our task here is to determine A and B. Multiply both sides by the least common denominator.

$$1 = A(s-2) + B(s-1)$$

Choose two random values of s to get two equations that we can use to solve for A and B.

$$\begin{aligned} s = 2: &\quad 1 = B(1) \\ s = 1: &\quad 1 = A(-1) \end{aligned}$$

The system yields $A = -1$ and $B = 1$, so the integral we have to evaluate in the exponent becomes

$$\int^x \left(-\frac{1}{s-1} + \frac{1}{s-2}\right) ds = -\ln(x-1) + \ln(x-2) = \ln\frac{x-2}{x-1}.$$

Hence,

$$I = e^{\ln\frac{x-2}{x-1}} = \frac{x-2}{x-1}.$$

Multiply both sides of the ODE by this integrating factor.

$$\frac{x-2}{x-1}y' + \frac{1}{(x-1)^2}y = \frac{2}{(x-1)^2}$$

The left side is now exact and can written as $d/dx(Iy)$ as a result of the product rule.

$$\frac{d}{dx}\left(\frac{x-2}{x-1}y\right) = \frac{2}{(x-1)^2}$$

Integrate both sides of the equation with respect to x.

$$\frac{x-2}{x-1}y = -\frac{2}{x-1} + C$$

Multiply both sides by $x-1$ and divide both sides by $x-2$ to solve for y.

$$y(x) = -\frac{2}{x-2} + \frac{C(x-1)}{x-2}$$

Combine the two terms into one.

$$y(x) = \frac{C(x-1)-2}{x-2}$$

Use the provided initial condition, $y(0) = 1$, to determine C.

$$1 = \frac{C(-1)-2}{-2} \quad \rightarrow \quad C = 0$$

Therefore,

$$y(x) = \frac{2}{2-x}.$$

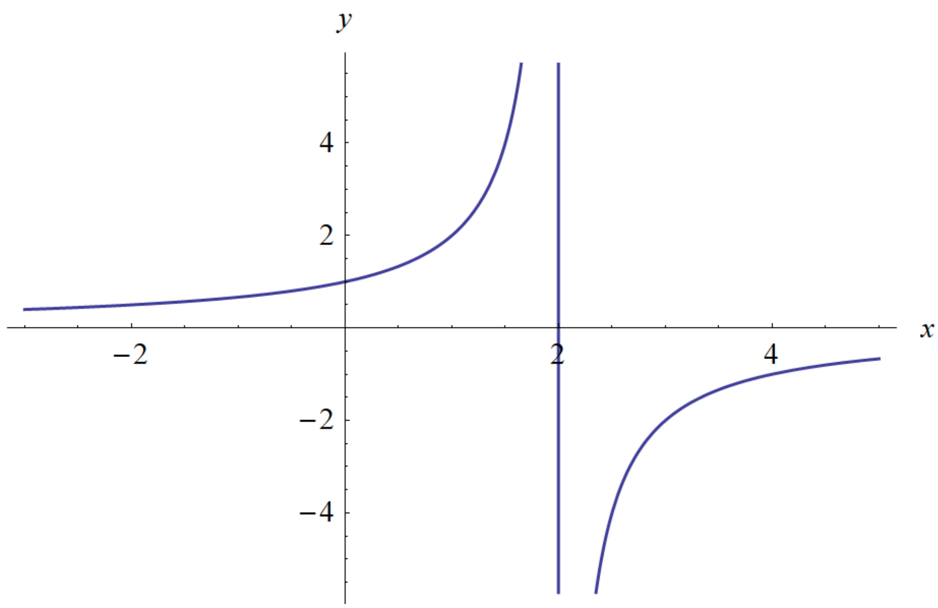

Figure 6: Plot of the solution for $-3 < x < 5$.

Part (y)

$$y' = 1/(x + e^y)$$

This ODE for $y(x)$ is quite difficult, so invert both sides of the equation.

$$\frac{dy}{dx} = \frac{1}{x + e^y} \quad \rightarrow \quad \frac{dx}{dy} = x + e^y$$

Bring x over to the left side.

$$\frac{dx}{dy} - x = e^y$$

This is a simpler first-order inhomogeneous ODE for x that can be solved with an integrating factor I. x is now the dependent variable, and y is now the independent variable.

$$I = e^{\int^y -1\,ds} = e^{-y}$$

Multiply both sides of the equation by I.

$$e^{-y}\frac{dx}{dy} - e^{-y}x = 1$$

The left side is now exact and can be written as $d/dy(Ix)$.

$$\frac{d}{dy}(e^{-y}x) = 1$$

Integrate both sides with respect to y.

$$e^{-y}x = y + C$$

Multiply both sides by e^y to solve for x.

$$x(y) = e^y(y + C)$$

This is an implicit solution for y.

Part (z)

$$xy' + y = y^2x^4$$

This is a Bernoulli equation. Start off by getting rid of the term multiplying y'. Divide both sides of the equation by x.

$$y' + \frac{1}{x}y = y^2x^3$$

Now divide both sides by y^2.

$$y^{-2}y' + \frac{1}{x}y^{-1} = x^3$$

Make the substitution,

$$u = y^{-1}$$

$$\frac{du}{dx} = (-1)y^{-2}\frac{dy}{dx} \quad \rightarrow \quad -\frac{du}{dx} = y^{-2}\frac{dy}{dx}.$$

Plug these expressions into the ODE.

$$-\frac{du}{dx} + \frac{1}{x}u = x^3$$

This is a first-order ODE that can be solved with an integrating factor. Multiply both sides by -1.

$$\frac{du}{dx} - \frac{1}{x}u = -x^3$$

The integrating factor is this.

$$I = e^{\int^x -\frac{1}{s}\,ds} = e^{-\ln x} = x^{-1}$$

Multiply both sides by I.

$$\frac{1}{x}\frac{du}{dx} - \frac{1}{x^2}u = -x^2$$

The left side is now exact and can be written as $d/dx(Iu)$ as a result of the product rule.

$$\frac{d}{dx}\left(\frac{1}{x}u\right) = -x^2$$

Integrate both sides with respect to x.

$$\frac{1}{x}u = -\frac{1}{3}x^3 + C$$

Multiply both sides to solve for u.

$$u(x) = -\frac{1}{3}x^4 + Cx$$

Change back now to the original variable y.

$$\frac{1}{y} = -\frac{1}{3}x^4 + Cx$$

Invert both sides to solve for y and then simplify the result.

$$y(x) = \frac{1}{-\frac{1}{3}x^4 + Cx} = \frac{3}{x(3C - x^3)} = \frac{3}{x(A - x^3)}$$